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The local superconvergence of the trilinear element for the three-dimensional Poisson problem ☆

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ABSTRACT

In this paper, we shall combine the finite element theory of Green's function presented in this paper, the extrapolation technique and the local symmetric technique to investigate the local superconvergence of the trilinear element for the three-dimensional Poisson equation.

1. Introduction

The superconvergent patch recovery technique, the local symmetric technique, the weak estimates and the integral identity technique all take important roles for the finite element superconvergence, which has been research focus for more than thirty years (see [1–9,11–15,17–36]). Assume that Π_k is the tensor-product interpolation operator of degree k. For the three-dimensional Poisson equation, Liu and Zhu (see [22,23]) proposed two weak estimates for tensor-product block finite elements of degree k and then obtained the superconvergent results as follows.

$$\left\|u^h - \Pi_k u\right\|_{W^{1,\infty}(\Omega)} \leqslant ch^{k+l}\|u\|_{W^{k+l+1,\infty}(\Omega)}, \tag{1.1}$$

where $l=1$ if $k=1$ and $l=2$ if $k \geqslant 2$. The local symmetric technique proposed by Schatz, Wahlbin, Sloan and Asadzadeh (see [1,2,13,25–27]) is also an important technique to study the local superconvergence of FEM for second-order elliptic problems in that the global smoothness of the accurate solution is not very high in nonsmooth domain (see [10]). Let ∂_l denote the directional derivative operator in the oriented l, and

$$\bar{\partial}_l v(x_0) = \frac{1}{2}\Big[\lim_{t\to+0} \partial_l v(x_0+tl) + \lim_{t\to+0} \partial_l v(x_0-tl)\Big]$$

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is said to be the averaged derivative in the direction l for v. Similarly, we can define the averaged gradient operator $\overline{\nabla}$. Assume that $u \in W^{k+2,\infty}(U_d(x_0))$ and x_0 is a local symmetry point which means that there exists a neighborhood $U_d(x_0)$ of x_0 satisfying the partitions covering it are symmetric with respect to the point x_0 (see [27]). They (see [1,2,13,25–27]) observed that, if k is even, then there exists $\delta = \frac{\tau}{s+n/q+1}$ and $d = h^\delta$ satisfying:

$$\left|(u-u^h)(x_0)\right| \leqslant c\left(\ln\frac{d}{h}h^{k+2}\|u\|_{W^{k+2,\infty}(U_d(x_0))} + d^{-t-\frac{N}{p}}\left\|u-u^h\right\|_{W_p^{-t}(\Omega)}\right), \tag{1.2}$$

if k is odd, then

$$\left|\overline{\nabla}(u-u^h)(x_0)\right| \leqslant c\left(h^{k+1}\|u\|_{W^{k+2,\infty}(U_d(x_0))} + d^{-1-t-\frac{N}{p}}\left\|u-u^h\right\|_{W_p^{-t}(\Omega)}\right). \tag{1.3}$$

Zhang and Ahmed (see [30,31]) presented a new finite element gradient recovery method and used the interior analysis to investigate the local superconvergence of it. There are many publications investigating extrapolation and splitting technique for finite element method [7,19,21].

He, Chen and Zhu (see [12]) combined the symmetry technique, the weak estimates and the integral identity technique to investigate the superconvergence of finite element method. Assume that odd $k \geqslant 3$, x_0 is an inner locally symmetric point for a uniform family of rectangular partitions $\mathcal{T}_h$. They improved the result of (1.1) as

$$\left|\overline{\nabla}(u^h - \Pi_k u)(x_0)\right| \leqslant c\left(h^{k+3} + h^{2k+2}\rho(x_0,\partial\Omega)^{-k}\right)|\ln h|^{\frac{4}{3}}\|u\|_{W^{k+4,\infty}(\Omega)}. \tag{1.4}$$

One observes from (1.1)–(1.4) that the weak estimates and the integral identity technique can be used to get higher superconvergence than the local symmetric technique and the latter does not demand that $\mathcal{T}_h$ is uniform and has lower demand for the global smoothness of the accurate solution. We observe that the extrapolation technique has hardly been used study the superconvergence for three-dimensional elliptic problem. Thus, on the basis of above consideration, we shall combine the weak estimates, the integral identity and extrapolation technique and the local symmetric technique to study the local superconvergence for the trilinear element. Let x_0, $U_d(x_0)$ be defined as above, x_0 be a nodal point for $\mathcal{T}_h$ and $\mathcal{T}_h$ be a uniform family of rectangular partitions on $U_d(x_0)$. We present the extrapolation technique for the trilinear element as follows. Let every partition contained in $U_d(x_0)$ be equally divided into eight partitions and every partition contained in $\Omega \setminus U_d(x_0)$ be divided into eight partitions, we obtain a family of partitions $\mathcal{T}_{h/2}$. Assume that $u^{\frac{h}{2}}$ is the finite element approximation of u over $\mathcal{T}_{h/2}$. Using the extrapolation technique, we obtain u_h^* by

$$u_h^* = \frac{4u^{\frac{h}{2}} - u^h}{3}. \tag{1.5}$$

The finite element theory of Green's function, the integral identity technique and the local symmetric technique are combined to obtain the following result.

Theorem 1.1. *Assume that $u \in W^{4,\infty}(U_d(x_0))$. Then*

$$\left|(u-u_h^*)(x_0)\right| \leqslant c\left(h^4|\ln h|^{\frac{7}{3}}\|u\|_{W^{4,\infty}(U_d(x_0))} + d^{-t-\frac{3}{2}}\left\|u-u^h\right\|_{H^{-t}(\Omega)}\right). \tag{1.6}$$

(1.6) *not only shows that the weak estimates and the integral identity technique can be used to find higher superconvergence than the local symmetric technique, but also shows that the local superconvergence can be obtained by the combination of the weak estimates, the integral identity technique and the local symmetric technique when the accurate solution u does not exist high global smoothness.*

The paper is organized as follows. In Section 2, we provide preliminaries which will be used in this paper. Section 3 will discuss the linear finite element theory for second-order elliptic problem in $\Re^3$. We shall give a proof of Theorem 1.1 on the basis of Section 3.

Notation. In this paper, some standard notation for the Sobolev spaces and their norms are used, and c denotes constants not necessarily the same at each occurrence, but always independent of h, u, $B(x,R) = \{y \in \Omega \mid |x-y| \leqslant R\}$, $S^h(B(x,R)) \subset S^h(\Omega)$, $\Pi_k u$ is the tensor-product block interpolation of degree k of u.

2. Preliminaries

Assume that $\Omega \subset \Re^3$ is a bounded domain. We shall consider the following Poisson problem

$$\begin{cases} Lu \equiv -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Assume that the bilinear form $A(\psi,\phi)$ is defined by

$$A(\psi,\phi)=\int_\Omega\left[\sum_{i,j=1}^{2}a_{ij}\frac{\partial\psi(x)}{\partial x_i}\frac{\partial\phi(x)}{\partial x_j}+\gamma\psi(x)\phi(x)\right]dx,\quad\forall\psi,\phi\in H^1(\Omega).\tag{2.2}$$

Then the weak form of (2.1) is to obtain $u\in H_0^1(\Omega)$ satisfy

$$A(u,v)=(f,v)\quad\forall v\in H_0^1(\Omega).$$

The finite element solution $u^h\in S_0^h\subset H_0^1(\Omega)$ such that

$$A\big(u^h,v\big)=(f,v)\quad\forall v\in S_0^h(\Omega).$$

Assume that E is a bounded domain and $\psi,\phi\in H^1(E)$. In this paper, we also use the following more general bilinear form $A_E(\psi,\phi)$

$$A_E(\psi,\phi)=\int_E\left[\sum_{i,j=1}^{2}a_{ij}\frac{\partial\psi(x)}{\partial x_i}\frac{\partial\phi(x)}{\partial x_j}+\gamma\psi(x)\phi(x)\right]dx.\tag{2.3}$$

Assume that $x_0\in\Omega$ and the Green's function $G_{x_0}(x)$ of (2.1) at point x_0 is defined by

$$A(G_{x_0},v)=v(x_0)\quad\forall v\in C_0^\infty(\Omega),\tag{2.4}$$

and the derivative Green's function $\partial_{x_0}G_{x_0}(x)$ is analogously defined by

$$A(\partial_{x_0}G_{x_0},v)=\partial v(x_0)\quad\forall v\in C_0^\infty(\Omega).\tag{2.5}$$

3. The linear finite element theory of Green's function for the problem (2.1)

First let us introduce some lemmas. The following lemma plays a critical role in our analysis.

Lemma 3.1. *(See [12].) Under the assumption that the boundary of Ω is smooth,*

$$\big\|\partial_{x_0}G_{x_0}-\partial_{x_0}G_{x_0}^h\big\|_{W^{1,\infty}(\Omega-B(x_0,R))}\leqslant ch^kR^{-k-3}|\ln h|^{\frac{4}{3}},\tag{3.1}$$

$$\big\|\partial_{x_0}G_{x_0}-\partial_{x_0}G_{x_0}^h\big\|_{L^\infty(\Omega-B(x_0,R))}\leqslant ch^kR^{-k-2}|\ln h|^{\frac{4}{3}}.\tag{3.2}$$

Now estimate $\|G_x-I_h^1G_x\|_{W^{1,1}(\Omega)}$.

Lemma 3.2. *Assume that $x\in e\in\mathcal{T}_h$ and $x_1,\dots,x_K$ are all nodal points of e. If there exists a constant c satisfying $\min_{1\leqslant l\leqslant M}|x-x_l|\geqslant ch$ and $\rho(x,\partial e)\geqslant ch$. Under the assumption that the boundary of Ω is smooth, then*

$$\big\|G_x-I_h^1G_x\big\|_{W^{1,1}(\Omega)}\leqslant ch|\ln h|.\tag{3.3}$$

Proof. One finds that $\|G_x-I_h^1G_x\|_{W^{1,1}(D)}$ can be split into

$$\big\|G_x-I_h^1G_x\big\|_{W^{1,1}(D)}=\big\|G_x-I_h^1G_x\big\|_{W^{1,1}(B(x,h))}+\big\|G_x-I_h^1G_x\big\|_{W^{1,1}(D\backslash B(x,h))}=I+II.\tag{3.4}$$

Note that (see [16]) $|\partial_x^\alpha\partial_y^\beta G_y(x)|\leqslant c|x-y|^{-|\alpha|-|\beta|-1}$ for any $|\alpha|+|\beta|>0$. One has

$$I\leqslant\|G_x\|_{W^{1,1}(B(x,h))}+\big\|I_h^1G_x\big\|_{W^{1,1}(B(x,h))}\leqslant ch+ch\leqslant ch,\tag{3.5}$$

$$II\leqslant ch\|G_x\|_{W^{2,1}(\Omega\backslash B(x,h))}\leqslant ch|\ln h|.\tag{3.6}$$

The combination of (3.5) and (3.6) gives the desired result (3.3). □

We now investigate $\|G_{x_0}-G_{x_0}^h\|_{W^{1,\infty}(\Omega\backslash B(x_0,R))}$ and $\|G_{x_0}-G_{x_0}^h\|_{L^\infty(\Omega\backslash B(x_0,R))}$.

Lemma 3.3. *Assume that $\Omega\subset\Re^3$ has a smooth boundary. Then, for the linear element,*

$$\big\|G_{x_0}-G_{x_0}^h\big\|_{W^{1,\infty}(\Omega\backslash B(x_0,R))}\leqslant chR^{-3}|\ln h|^{\frac{4}{3}},\tag{3.7}$$

$$\big\|G_{x_0}-G_{x_0}^h\big\|_{L^\infty(\Omega\backslash B(x_0,R))}\leqslant ch^2R^{-3}|\ln h|^{\frac{4}{3}}.\tag{3.8}$$

Proof. Define the derivative Green's function $\partial_x G_x(y)$ by $A(\partial_x G_x, v) = \partial v(x)\ \forall v \in C_0^\infty(\Omega)$. Let $g_x(y) = \partial_x G_x(y)$, and let $g_x^h(y)$ be the finite element approximation of $g_x(y)$ over $\mathcal{T}_h$. Note that

$$\partial\big(G_{x_0} - G_{x_0}^h\big)(x) = A\big(G_{x_0} - G_{x_0}^h, g_x\big) = A\big(G_{x_0} - I_h^1 G_{x_0}, g_x - g_x^h\big) = A\big(G_{x_0}, g_x - g_x^h\big) = \big(g_x - g_x^h\big)(x_0).$$

This gives the desired result (3.7). We turn now to the estimation of (3.8). Assume that $x \in e_x \subset \Omega \setminus B(x_0, R)$ and the bilinear form $A_E(\psi, \phi)$ is defined by (2.3). One has

$$\begin{aligned}\big(G_{x_0} - G_{x_0}^h\big)(x) &= A\big(G_{x_0} - I_h^1 G_{x_0}, G_x - G_x^h\big)\\ &= A_{B(x_0, \frac{R}{2})}\big(G_{x_0} - I_h^1 G_{x_0}, G_x - G_x^h\big) + A_{\Omega\setminus B(x_0, \frac{R}{2})}\big(G_{x_0} - I_h^1 G_{x_0}, G_x - G_x^h\big) = I_1 + I_2.\end{aligned}$$

The combination of Lemmas 3.1 and 3.2 gives

$$|I_1| \leqslant c\big\|G_x - I_h^1 G_x\big\|_{W^{1,1}(\Omega)}\big\|G_x - G_x^h\big\|_{W^{1,\infty}(\Omega\setminus B(x, \frac{R}{2}))} \leqslant ch^2 R^{-3}|\ln h|^{\frac{4}{3}}. \tag{3.9}$$

Now estimate I_2. Assume that $\phi(y)$ is a scalar function in C^∞ such that $\phi(y) = 1$ if $y \in B(x, \frac{R}{4})$, $\phi(y) = 0$ if $y \in \Omega \setminus B(x, \frac{R}{2})$, $|\nabla\phi|_{L^\infty(\Omega)} \leqslant cR^{-1}$. Let $\psi_1(y) = \phi(y)(G_x - I_h^1 G_x)(y)$ and $\psi_2(y) = [1 - \phi(y)](G_x - I_h^1 G_x)(y)$. By Lemmas 3.1 and 3.2,

$$\begin{aligned}|I_2| &\leqslant \big|A_{\Omega\setminus B(x_0, \frac{R}{2})}\big(\psi_1, G_x - G_x^h\big)\big| + \big|A_{\Omega\setminus B(x_0, \frac{R}{2})}\big(\psi_2, G_x - G_x^h\big)\big|\\ &= \big|A\big(\psi_1, G_x - G_x^h\big)\big| + \big|A_{\Omega\setminus B(x_0, \frac{R}{2})}\big(\psi_2, G_x - G_x^h\big)\big|\\ &\leqslant ch^2\|\psi_1\|_{W^{2,\infty}(\Omega)} + \big\|G_x - G_x^h\big\|_{W^{1,\infty}(\Omega\setminus B(x, \frac{R}{4}))}\|\psi_2\|_{W^{1,1}(\Omega)}\\ &\leqslant ch^2 R^{-3} + chR^{-3}|\ln h|^{\frac{4}{3}}h \leqslant ch^2 R^{-3}|\ln h|^{\frac{4}{3}}.\end{aligned}$$

This, together with (3.9), gives the desired result (3.8). This completes the proof. □

We are now in a position to give the main results of this section.

Theorem 3.1. *Under the assumptions of Lemma* 3.3*,*

$$\big|G_{x_0} - G_{x_0}^h\big|_{L^1(\Omega)} \leqslant ch^2|\ln h|^{\frac{7}{3}}, \tag{3.10}$$

$$\sum_{e\in\mathcal{T}_h}\big\|G_{x_0}^h\big\|_{W^{2,1}(e)} \leqslant c|\ln h|^{\frac{4}{3}}. \tag{3.11}$$

Proof. Split $|G_{x_0} - G_{x_0}^h|_{L^1(\Omega)}$ into

$$\big|G_{x_0} - G_{x_0}^h\big|_{L^1(\Omega)} = \big|G_{x_0} - G_{x_0}^h\big|_{L^1(B(x_0,2h))} + \big|G_{x_0} - G_{x_0}^h\big|_{L^1(\Omega\setminus B(x_0,2h))} = I_1 + I_2. \tag{3.12}$$

First estimate I_1. One finds that

$$c\big\|G_{x_0}^h\big\|_{L^\infty(\Omega)} \leqslant ch^{\frac{-1}{2}}\big\|G_{x_0}^h\big\|_{H^1(\Omega)} \leqslant ch^{\frac{-1}{2}}\big[A\big(G_{x_0}^h, G_{x_0}^h\big)\big]^{\frac{-1}{2}} \leqslant ch^{\frac{-1}{2}}\big[\big\|G_{x_0}^h\big\|_{L^\infty(\Omega)}\big]^{\frac{-1}{2}}. \tag{3.13}$$

Consequently,

$$I_1 \leqslant \|G_{x_0}\|_{L^1(B(x_0,R))} + \big\|G_{x_0}^h\big\|_{L^1(B(x_0,R))} \leqslant ch^2 + ch^3\big\|G_{x_0}^h\big\|_{L^\infty(\Omega)} \leqslant ch^2. \tag{3.14}$$

Note that (3.8) gives

$$I_2 \leqslant ch^2|\ln h|^{\frac{7}{3}}. \tag{3.15}$$

Hence, by (3.12), (3.14) and (3.15),

$$\big\|G_{x_0} - G_{x_0}^h\big\|_{L^1(\Omega)} \leqslant ch^2|\ln h|^{\frac{7}{3}}.$$

Similarly,

$$\big\|G_{x_0} - G_{x_0}^h\big\|_{W^{1,1}(\Omega)} \leqslant ch|\ln h|^{\frac{7}{3}}. \tag{3.16}$$

We turn now to the proof of (3.16). One has

$$\begin{aligned}\sum_{e\in\mathcal{T}_h}\|G^h_{x_0}\|_{W^{2,1}(e)} &= \sum_{e\subset B(x_0,2h)}\|G^h_{x_0}\|_{W^{2,1}(e)} + \sum_{e\subset\Omega\setminus B(x_0,2h)}\|G^h_{x_0}\|_{W^{2,1}(e)}\\
&\leqslant ch^3h^{-2}h^{-1} + \sum_{e\subset\Omega\setminus B(x_0,2h)}\|G^h_{x_0}-G^I_{x_0}\|_{W^{2,1}(e)} + \sum_{e\subset\Omega\setminus B(x_0,2h)}\|G^I_{x_0}\|_{W^{2,1}(e)}\\
&\leqslant c + ch^{-1}\big(\|G^h_{x_0}-G_{x_0}\|_{W^{1,1}(\Omega\setminus B(x_0,2h))} + \|G_{x_0}-G^I_{x_0}\|_{W^{1,1}(\Omega\setminus B(x_0,2h))}\big) + c|\ln h|\\
&\leqslant ch^{-1}h|\ln h|^{\frac{4}{3}} + c|\ln h| \leqslant c|\ln h|^{\frac{4}{3}}.\end{aligned}$$

This ends the proof. □

The following result can be deduced from Theorem 3.1, directly.

Corollary 3.1. *There exists a constant c such that*

$$\big|G_{x_0}-G^h_{x_0}\big|_{L^1(\Omega)} \leqslant ch^2|\ln h|^{\frac{7}{3}}, \tag{3.17}$$

$$\sum_{e\in\mathcal{T}_h}\|G^h_{x_0}\|_{W^{2,1}(e)} \leqslant c|\ln h|^{\frac{4}{3}}. \tag{3.18}$$

Proof. Assume that $\Omega_1 \supset \Omega$ has a smooth boundary and $A_{\Omega_1}(\psi,\phi)$ is defined as (2.3). Define G_1 by

$$A_{\Omega_1}(G_1, v) = v(x_0) \quad \forall v\in C_0^\infty(\Omega_1). \tag{3.19}$$

Let $\hat{\mathcal{T}}_h$ be a family of partitions on Ω_1 with grid size h satisfying $e\in\hat{\mathcal{T}}_h$ whenever $e\in\mathcal{T}_h$ and R_hG_1 is the finite element approximation of G_1 over $\hat{\mathcal{T}}_h$. Let $E_G = G_1 - G_{x_0}$ and E^h_G be the finite element approximation of E_G over $\mathcal{T}_h$. One observes that $G_{x_0}-G^h_{x_0}$ can be split into

$$G_{x_0}-G^h_{x_0} = (G_1 - R_hG_1) + \big(E_G - E^h_G\big) + \big(R_hG_1 - G^h_1\big) = I_1(x) + I_2(x) + I_3(x). \tag{3.20}$$

One observes that

$$\|I_1\|_{L^1(\Omega)} \leqslant ch^2|\ln h|^{\frac{7}{3}}, \qquad \|I_3\|_{L^1(\Omega)} \leqslant ch^2\|E_G\|_{H^2(\Omega)} \leqslant ch^2. \tag{3.21}$$

Note that Theorem 3.1 gives

$$\|I_2\|_{L^1(\Omega)} \leqslant c\|I_2\|_{L^\infty(\Omega)} \leqslant c\|I_2\|_{L^\infty(\partial\Omega)} \leqslant ch^2|\ln h|^{\frac{7}{3}}. \tag{3.22}$$

From (3.20), (3.21) and (3.22) it follows that

$$\big|G_{x_0}-G^h_{x_0}\big|_{L^1(\Omega)} \leqslant ch^2|\ln h|^{\frac{7}{3}}.$$

Similarly, we get the desired result (3.18). □

4. Proof of Theorem 1.1

By the linear FEM theory of Green's function presented in the Section 3, we will give a proof of Theorem 1.1. First we discuss some lemmas. Assume that Ω is a rectangular block with boundary $\partial\Omega$ and $\mathcal{T}_h$ is a uniform family of rectangular partitions with grid size h. Set

$$D_1 = \frac{\partial}{\partial x_1}, \qquad D_2 = \frac{\partial}{\partial x_2}, \qquad D_3 = \frac{\partial}{\partial x_3}.$$

Further, for any $e\in\mathcal{T}_h$, let $x_e = (x_{1,e}, x_{2,e}, x_{3,e})$ be the center of e. Assume that $x = (x_1, x_2, x_3)$. For $1\leqslant i\leqslant 3$, set

$$B(x_i) = \frac{1}{2}\big[(x_i - x_{i,e})^2 - h^2\big], \qquad F(x_i) = \frac{1}{6}B^2(x_i).$$

By the integral identity technique presented by Lin et al. (see [19,21]), we get the following result.

Lemma 4.1. *Assume that $u \in W^{4,\infty}(\Omega) \cap H_0^1(\Omega)$, and for any $e \in \mathcal{T}_h$, $u \in W^{5,\infty}(e)$. Then, for any $v \in S_0^h(\Omega)$,*

$$\begin{aligned}
A\big(u-u^I, v\big) &= h^2 \sum_{e\in\mathcal{T}_h} \int_e \big(D_1^2D_2^2u + D_1^2D_3^2u + D_2^2D_3^2u\big)(x)v(x)\,dx \\
&\quad + \sum_{e\in\mathcal{T}_h} \int_e F(x_1)\big(-D_1^4u(x)D_2^2v(x) + 4D_1^3D_2u(x)D_1D_2v(x)\big)\,dx \\
&\quad + \sum_{e\in\mathcal{T}_h} \int_e F(x_2)\big(-D_2^4u(x).D_3^2v(x) + 4D_2^3D_3u(x)D_2D_3v(x)\big)\,dx \\
&\quad + \sum_{e\in\mathcal{T}_h} \int_e F(x_3)\big(-D_3^4u(x)D_1^2v(x) + 4D_3D_1u(x)D_1D_3v(x)\big)\,dx. \qquad (4.1)
\end{aligned}$$

Proof. One has, for any $v \in S_0^h(\Omega)$,

$$\begin{aligned}
\int_e D_1\big(u-u^I\big)(x)D_1v(x)\,dx &= -\int_e B'(x_3)D_3\big(D_1\big(u-u^I\big)(x)D_1v(x)\big)\,dx \\
&= \int_e B(x_3)D_3^2\big(D_1\big(u-u^I\big)(x)D_1v(x)\big)\,dx \\
&= \int_e B(x_3)D_3^2D_1u(x)D_1v(x)\,dx + 2\int_e B(x_3)D_1D_3\big(u-u^I\big)(x)D_1D_3v(x)\,dx \\
&= I_1 + I_2. \qquad (4.2)
\end{aligned}$$

Note that

$$\begin{aligned}
I_1 &= \int_e \big[F''(x_3) - h^2/3\big]D_3^2D_1u(x)D_1v(x)\,dx \\
&= \int_e F(x_3)D_3^2\big[D_3^2D_1u(x)D_1v(x)\big]\,dx - \frac{h^2}{3}\int_e D_3^2D_1u(x)D_1v(x)\,dx \\
&= \int_e F(x_3)\big[D_3^4D_1u(x)D_1v(x) + 2D_3^3D_1u(x)D_1D_3v(x)\big]\,dx - \frac{h^2}{3}\int_e D_3^2D_1u(x)D_1v(x)\,dx, \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= 2\int_e F''(x_3)D_1D_3\big(u-u^I\big)(x)D_1D_3v(x)\,dx - \frac{2h^2}{3}\int_e D_1D_3\big(u-u^I\big)(x)D_1D_3v(x)\,dx \\
&= -2\int_e F'(x_3)D_1D_3^2u(x)D_1D_3v(x)\,dx = 2\int_e F(x_3)D_1D_3^3u(x)D_1D_3v(x)\,dx. \qquad (4.4)
\end{aligned}$$

By (4.2), (4.3) and (4.4),

$$\begin{aligned}
&\int_e D_1\big(u-u^I\big)(x)D_1v(x)\,dx \\
&\quad = -\frac{h^2}{3}\int_e D_3^2D_1u(x)D_1v(x)\,dx + 4\int_e F(x_3)D_3^3D_1u(x).D_1D_3v(x)\,dx + \int_e F(x_3)D_3^4D_1u(x).D_1v(x)\,dx. \qquad (4.5)
\end{aligned}$$

Note that

$$\sum_{e\in\mathcal{T}_h}\int_e F(x_3)D_3^4D_1u(x).D_1v(x)\,dx = -\sum_{e\in\mathcal{T}_h}\int_e F(x_3)D_3^4u(x).D_1^2v(x)\,dx. \qquad (4.6)$$

The combination of (4.5) and (4.6) gives

$$\int_\Omega D_1\big(u-u^I\big)(x)D_1v(x)\,dx$$
$$=-\frac{h^2}{3}\int_\Omega D_3^2D_1u(x)D_1v(x)\,dx+4\sum_{e\in\mathcal{T}_h}\int_e F(x_3)D_3^3D_1u(x)D_1D_3v(x)\,dx+\sum_{e\in\mathcal{T}_h}\int_e F(x_3)D_3^4D_1u(x)D_1v(x)\,dx$$
$$=\frac{h^2}{3}\int_\Omega D_3^2D_1^2u(x)v(x)\,dx+4\sum_{e\in\mathcal{T}_h}\int_e F(x_3)D_3^3D_1u(x)D_1D_3v(x)\,dx-\sum_{e\in\mathcal{T}_h}\int_e F(x_3)D_3^4u(x)D_1^2v(x)\,dx. \tag{4.7}$$

Similarly,

$$\int_\Omega D_2\big(u-u^I\big)(x)D_2v(x)\,dx$$
$$=\frac{h^2}{3}\int_\Omega D_2^2D_1^2u(x)v(x)\,dx+4\sum_{e\in\mathcal{T}_h}\int_e F(x_1)D_1^3D_2u.D_1D_2v(x)]\,dx-\sum_{e\in\mathcal{T}_h}\int_e F(x_1)D_1^4u(x).D_2^2v(x)\,dx, \tag{4.8}$$

and

$$\int_\Omega D_3\big(u-u^I\big)(x)D_3v(x)\,dx$$
$$=\frac{h^2}{3}\int_\Omega D_2^2D_3^2u(x)v(x)\,dx+4\sum_{e\in\mathcal{T}_h}\int_e F(x_2)D_2^3D_3u(x)D_2D_3v(x)\,dx-\sum_{e\in\mathcal{T}_h}\int_e F(x_2)D_2^4u(x)D_3^2v(x)\,dx. \tag{4.9}$$

The combination of (4.7), (4.8) and (4.9) gives the desired result (4.1). □

On the basis of Theorem 3.1 and Lemma 4.1, we have the following superconvergent result.

Lemma 4.2. *Assume that u_h^* is defined as* (1.5)*. Under the assumptions of Lemma* 4.1*,*

$$\big|\big(u_h^*-u\big)(x_0)\big|\leqslant ch^4|\ln h|^{\frac{7}{3}}\|u\|_{W^{4,\infty}(\Omega)}. \tag{4.10}$$

Proof. Note that x_0 is a nodal point. One observes that

$$\big(u^h-u\big)(x_0)=\big(u^h-u^I\big)(x_0)=A\big(u^h-u^I,G_{x_0}^h\big)=A\big(u-u^I,G_{x_0}^h\big). \tag{4.11}$$

From the combination of Lemma 4.1 and (4.11), one observes that $(u^h-u)(x_0)$ can be decomposed into

$$\begin{aligned}\big(u^h-u\big)(x_0)=h^2&\sum_{e\in\mathcal{T}_h}\int_e\big(D_1^2D_2^2u+D_1^2D_3^2u+D_2^2D_3^2u\big)(x)G_{x_0}^h(x)\,dx\\&+\sum_{e\in\mathcal{T}_h}\int_e F(x_1)\big(-D_1^4u(x)D_2^2G_{x_0}^h(x)+4D_1^3D_2u(x)D_1D_2G_{x_0}^h(x)\big)\,dx\\&+\sum_{e\in\mathcal{T}_h}\int_e F(x_2)\big(-D_2^4u(x).D_3^2G_{x_0}^h(x)+4D_2^3D_3u(x)D_2D_3G_{x_0}^h(x)\big)\,dx\\&+\sum_{e\in\mathcal{T}_h}\int_e F(x_3)\big(-D_3^4u(x)D_1^2G_{x_0}^h(x)+4D_3^3D_1u(x)D_1D_3G_{x_0}^h(x)\big)\,dx=I_1+I_2+I_3+I_4.\end{aligned} \tag{4.12}$$

First estimate I_1. One observes that I_1 can be split into

$$\begin{aligned}I_1={}&h^2\sum_{e\in\mathcal{T}_h}\int_e\big(D_1^2D_2^2u+D_1^2D_3^2u+D_2^2D_3^2u\big)(x)G_{x_0}(x)\,dx\\&+h^2\sum_{e\in\mathcal{T}_h}\int_e\big(D_1^2D_2^2u+D_1^2D_3^2u+D_2^2D_3^2u\big)(x)\big(G_{x_0}^h(x)-G_{x_0}(x)\big)\,dx=I_{1,1}+I_{1,2}.\end{aligned} \tag{4.13}$$

This implies that there exists a constant d, independent of h, such that

$$I_{1,1} = dh^2. \tag{4.14}$$

Note that (3.17) gives

$$|I_{1,2}| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(\Omega)}. \tag{4.15}$$

Hence by (4.14) and (4.15),

$$\left|I_1 - dh^2\right| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(\Omega)}. \tag{4.16}$$

Now estimate I_2, I_3 and I_4, respectively. The combination of (3.18) and (4.12) gives

$$|I_2| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(\Omega)}, \tag{4.17}$$

$$|I_3| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(\Omega)}, \tag{4.18}$$

$$|I_4| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(\Omega)}. \tag{4.19}$$

One concludes from (4.12), (4.13), (4.16), (4.17), (4.18) and (4.19) that

$$\left|\left(u^h - u\right)(x_0) - dh^2\right| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(\Omega)}. \tag{4.20}$$

This gives

$$\left|\left(u_h^* - u\right)(x_0)\right| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(\Omega)}. \tag{4.21}$$

The proof is complete. $\square$

On the basis of Lemma 4.2, we are now in a position to give a proof of Theorem 1.1.

Let $D \subset \Omega$ be a rectangular block such that $\rho(\partial\Omega, \partial D) \geqslant c$, $\overline{\mathcal{T}}_h$ is a uniform family of rectangular partitions on D satisfying, for any $e \in \mathcal{T}_h$, $e \in \overline{\mathcal{T}}_h$ whenever $e \subset U_d(x_0)$. Assume that $u_1(x) = u(x)$ if $x \in U_{d/2}(x_0)$, $\|u_1\|_{W^{4,\infty}(D)} \leqslant c\|u\|_{W^{4,\infty}(U_d(x_0))}$, $u_1|_{\partial D} = 0$. Let $R_h u_1(x)$ and $R_{h/2} u_1(x)$ be the tensor-product block linear element approximations of $\overline{\mathcal{T}}_h$ and $\overline{\mathcal{T}}_{h/2}$, respectively, where $\overline{\mathcal{T}}_h$ is obtained by dividing each partition of $\overline{\mathcal{T}}_h$ into eight rectangular partitions equally. Set

$$R_{h,*} u_1(x) = \frac{4R_{h/2} u_1(x) - R_h u_1(x)}{3}. \tag{4.22}$$

We observe that $(u_*^h - u)(x_0)$ can be decompose into

$$\left(u_*^h - u\right)(x_0) = (R_{h,*} u_1 - u_1)(x_0) + \left(u_*^h - R_{h,*} u_1\right)(x_0) = I_1 + I_2. \tag{4.23}$$

Note that Theorem 4.1 gives

$$|I_1| \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u_1\|_{W^{4,\infty}(D)} \leqslant ch^4 |\ln h|^{\frac{7}{3}} \|u\|_{W^{4,\infty}(U_d(x_0))}, \tag{4.24}$$

and

$$\begin{aligned} \left|\left(u^h - R_h u_1\right)(x_0)\right| &\leqslant \left|u^h - R_h u_1\right|_{L^\infty(U_{d/2}(x_0))} \leqslant cd^{-t-\frac{3}{2}} \left\|u^h - R_h u_1\right\|_{H^{-t}(U_d(x_0))} \\ &\leqslant cd^{-t-\frac{3}{2}} \left(\left\|u^h - u\right\|_{H^{-t}(U_d(x_0))} + \|u_1 - R_h u_1\|_{H^{-t}(U_d(x_0))}\right) \leqslant cd^{-t-\frac{3}{2}} \left\|u^h - u\right\|_{H^{-t}(\Omega)}, \end{aligned}$$

which implies

$$|I_2| \leqslant cd^{-t-\frac{3}{2}} \left\|u - u^h\right\|_{H^{-t}(\Omega)}. \tag{4.25}$$

The desired result (1.6) can be deduced from the combination of (4.24) and (4.25).

5. Numerical example

In this section, we shall use numerical examples to investigate Theorem 1.1. Consider the Poisson problem in a square $\Omega = (0,1)^3$,

$$\begin{cases} \Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

where $f(x) = 2\sum_{i=1}^{3} \prod_{i=1, i\neq j}^{3} x_i(1 - x_i)$.

Table 1
Numerical experiment for the trilinear element.

	$h=\frac{1}{5}$	$h=\frac{1}{6}$	$h=\frac{1}{7}$	$h=\frac{1}{8}$	$h=\frac{1}{9}$	$h=\frac{1}{10}$
$\Vert u_h^*-u\Vert_{\infty,h}$	1.09×10^{-5}	5.60×10^{-6}	3.64×10^{-6}	1.72×10^{-6}	8.01×10^{-8}	6.96×10^{-7}
$\frac{\Vert u_h^*-u\Vert_{\infty,h}}{h^4}$	6.84×10^{-3}	7.25×10^{-3}	8.74×10^{-3}	7.05×10^{-3}	5.25×10^{-3}	6.96×10^{-3}
$\frac{\Vert u_h^*-u\Vert_{\infty,h}}{h^4\vert\ln h\vert^{\frac{7}{3}}}$	2.3×10^{-3}	1.9×10^{-3}	1.8×10^{-3}	1.3×10^{-3}	8.37×10^{-4}	9.94×10^{-4}

The exact solution $u(x)=x_1(1-x_1)x_2(1-x_2)x_3(1-x_3)$. Let $\mathcal{T}_h$ be the uniform family of rectangular partitions with grid size h. Assume that u_h^* is defined as in Section 1 and $M=\{x_1,\dots,x_N\}$ is a set composed of all nodal points of $\mathcal{T}_h$. Set

$$\left\Vert u_h^*-u\right\Vert_{\infty,h}=\max_{x_i\in M}\left|\left(u-u_h^*\right)(x_i)\right|. \tag{5.2}$$

We have the following Table 1.

The above numerical example shows that there exists a constant c such that

$$\left\Vert u_h^*-u\right\Vert_{\infty,h}\leqslant ch^4|\ln h|^{\frac{7}{3}}. \tag{5.3}$$

which implies that Theorem 1.1 is valid.

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